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Optimal Die Design for Work-Hardening Metals

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This paper discusses a die design problem for rigid-plastic metals under plane strain conditions. The strain distribution in the end product is required to be uniform, a condition which is satisfied when the principal stress directions coincide with streamlines of the flow. All possible deformations corresponding to a linear work hardening rule will be described. © 1986 Academic Press, Inc.

1. INTRODUCTION

The equations describing a steady state deformation of an ideal perfectly plastic solid are known to be hyperbolic (see Hill [3], Johnson [6], and Hopkins [5]) and thus they are most conveniently written in characteristic coordinates. The characteristic curves are commonly called α- and β-sliplines, and the two families are everywhere orthogonal, each making a 45-degree angle with the major principal stress direction. The equations in α-, β-coordinates are

$$\frac{\partial p}{\partial s_\alpha} + 2k\frac{\partial \phi}{\partial s_\alpha} = 0 \tag{1.1}$$

$$\frac{\partial p}{\partial s_\beta} - 2k\frac{\partial \phi}{\partial s_\beta} = 0 \tag{1.2}$$

$$\frac{\partial u}{\partial s_\alpha} - v\frac{\partial \phi}{\partial s_\alpha} = 0 \tag{1.3}$$

$$\frac{\partial v}{\partial s_\beta} + u\frac{\partial \phi}{\partial s_\beta} = 0. \tag{1.4}$$

Here p is the hydrostatic part of the stress tensor

$$\Sigma = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix}$$

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and is often called the (hydrostatic) pressure,

$$p = -\frac{1}{2}(\sigma_x + \sigma_y).$$

$\partial/\partial s_\alpha$ and $\partial/\partial s_\beta$ refer to the differentiation along the characteristics and ϕ is the angle of inclination of α-lines with respect to a fixed reference direction. u and v denote velocity components along α- and β-directions, respectively. The yield condition hidden in Eq. (1.1)–(1.4) is

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy} = 4k^2 \tag{1.5}$$

where k is taken to be a known constant.

The derivation of Eqs. (1.1)–(1.4) includes several simplifications: the elastic behavior of the material and inertial terms due to acceleration are ignored. It is generally accepted that in situations where stresses are large, e.g., in most metal-forming processes, the approximations are quite valid. However, as metal deforms its resistance for further deformations increases and k cannot be regarded as a constant throughout the flow. Rather, for work-hardening metals, k should be regarded as one of the unknown functions. Equations (1.1) and (1.2) must be rewritten

$$\frac{\partial p}{\partial s_\alpha} + 2k\frac{\partial \phi}{\partial s_\alpha} - \frac{\partial k}{\partial s_\beta} = 0 \tag{1.6}$$

$$\frac{\partial p}{\partial s_\beta} - 2k\frac{\partial \phi}{\partial s_\beta} - \frac{\partial k}{\partial s_\alpha} = 0 \tag{1.7}$$

and we must add another equation describing the evolution of k (derived by Richmond, see Devenpeck and Weinstein [2]),

$$u\frac{\partial k}{\partial s_\alpha} + v\frac{\partial k}{\partial s_\beta} = h(2\gamma) \tag{1.8}$$

where

$$\gamma = \frac{1}{2}\left(\frac{\partial v}{\partial s_\alpha} + u\frac{\partial \phi}{\partial s_\alpha} + \frac{\partial u}{\partial s_\beta} - v\frac{\partial \phi}{\partial s_\beta}\right). \tag{1.9}$$

In this equation γ is the shear strain-rate and the function h is called the hardening modulus.

Quite a lot is known about solutions of Eq. (1.1)–(1.4) in connection with various metal-deforming processes. However, the theory of deformations taking place in hardening materials is an interesting new area to study. In the literature there appears to be only one solution known to Eqs. (1.3), (1.4), and (1.6)–(1.9) (see Collins [1]).

We will study this system under the hypothesis that the velocity and stress fields satisfy an optimality condition described in the next section. Section 3 contains a derivation of a streamline-material line coordinate system which is used in the following section to find the general solution of the problem. In Section 5, we develop some properties of solutions which allow us to sketch the resulting die profiles in Section 6. The last section is devoted to the difficulty of finding complete solutions, i.e., solutions which can be joined smoothly to rigid regions on both sides of the die.

2. Ideal Dies

Richmond [7, 8] has introduced the concept of ideal dies for drawing processes. He requires that the streamlines coincide with one family of principal stress trajectories and shows that in this case the final deformation is uniform and no redundant work is done in the deformation. Hill [4] has also discussed flows with this property.

Let us now give a proof of a fact that ideal flows lead to uniform deformations. Consider a uniform grid consisting of streamlines and lines perpendicular to them in the material entering the die. These perpendicular lines will be called material lines. The goal is to show that the orthogonality persists while the material deforms travelling through the die. This is a statement about the velocity field $\mathbf{u}$ only. More exactly, we have

LEMMA. *Let δ be a line everywhere orthogonal to streamlines. The orthogonality of δ to streamlines is preserved under the flow iff*

$$\nabla x \left(\frac{\mathbf{u}}{|u|^2}\right) = \mathbf{0}. \tag{2.1}$$

Proof. Let γ_1 and γ_2 be two streamlines and Ω the region bounded by them and two material lines δ_1 and δ_2 perpendicular to streamlines everywhere. See Fig. 1.

Let T_{AB} denote the time it takes a particle to travel from point A to point B. Then

$$\begin{aligned} T_{AB} - T_{CD} &= \int_{\gamma_1 \cap \partial\Omega} \frac{ds}{|u|} - \int_{\gamma_2 \cap \partial\Omega} \frac{ds}{|u|} \\ &= -\int_{\partial\Omega} \frac{\mathbf{u} \cdot \mathbf{d}s}{|u|^2} \\ &= -\int_{\Omega} \nabla x \left(\frac{\mathbf{u}}{|u|^2}\right) \cdot \mathbf{k}\, dx\, dy. \end{aligned}$$

Here $\mathbf{k}$ is the unit vector perpendicular to the plane of flow.

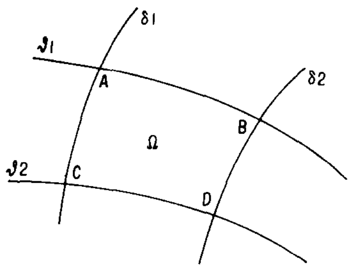

FIGURE 1

Requiring that section BD be the image of AC under the flow is then equivalent to (2.1). ▮

Transforming condition (2.1) into characteristic coordinates gives

$$\nabla x \left(\frac{\mathbf{u}}{|u|^2}\right) \cdot \mathbf{k} = 2\gamma \frac{u^2 - v^2}{(u^2 + v^2)^2} = 0$$

$$\Rightarrow u = \pm v.$$

Recalling that characteristics make 45-degree angles with principal stress directions, we get Richmond's condition that flow is everywhere in the direction of one of the principal stresses. Note that this conclusion holds equally for work-hardening and non-work-hardening materials.

3. (T, L)-COORDINATES

The orthogonal grid described above can be used as a basis for an orthogonal coordinate system. In the case $u = v$, Eqs. (1.3), (1.4), and (1.6)–(1.9) reduce into form

$$\frac{\partial p}{\partial s_\alpha} + 2k \frac{\partial (\ln u)}{\partial s_\alpha} - \frac{\partial k}{\partial s_\beta} = 0 \tag{3.1}$$

$$\frac{\partial p}{\partial s_\beta} + 2k \frac{\partial (\ln u)}{\partial s_\beta} - \frac{\partial k}{\partial s_\alpha} = 0 \tag{3.2}$$

$$\frac{\partial k}{\partial s_\gamma} = \frac{1}{\sqrt{2}\, u} h\left(2\sqrt{2} \frac{\partial u}{\partial s_\gamma}\right) \tag{3.3}$$

$$\frac{\partial}{\partial s_\alpha} (\ln u - \phi) = 0 \tag{3.4}$$

$$\frac{\partial}{\partial s_\beta} (\ln u + \phi) = 0, \tag{3.5}$$

where $\partial/\partial s_\gamma = (1/\sqrt{2})(\partial/\partial s_\alpha + \partial/\partial s_\beta)$ denotes differentiation in the direction of streamlines. This system is overdetermined, for four unknown functions we have five equations. The equations split naturally into two groups: the first three stress equations form a system for three unknowns p, k and u and the last two velocity equations for ϕ and u. Both systems are easily shown to be hyperbolic, actually (3.4) and (3.5) are already in characteristic form. The characteristics of system (3.1)–(3.3) turn out to be streamlines and material lines. Streamlines are double characteristics. Equation (3.3) already involves differentiations in the direction of streamlines only. To write the first two equations in characteristic form we need a new coordinate system. Let these coordinates be T and L, where $L=$ constant gives equations of streamlines and $T=$ constant gives material lines. T denotes the time a particle has travelled from an arbitrarily chosen initial material line. L gives the distance of a streamline from the side of the die along the initial material line $T=0$. According to Section 2 these coordinates form an orthogonal system. Figure 2 illustrates (T, L)-coordinates for a symmetric die . The velocity at $T=0$, $u(0, L)=u_0$ would be a constant in this die but generally u_0 is a function of L. Later on we will give examples of these types of dies. Notice that the grid formed by lines $L=n\Delta L$ and $T=m\Delta T$ divides the die into regions of equal area. This is a consequence of the incompressibility of the material.

The derivatives along the characteristics transform according to formulas

$$\frac{\partial}{\partial s_\gamma} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial s_\alpha} + \frac{\partial}{\partial s_\beta}\right) = \frac{1}{\sqrt{2}\,u}\frac{\partial}{\partial T} \tag{3.6}$$

$$\frac{\partial}{\partial s_\delta} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial s_\alpha} - \frac{\partial}{\partial s_\beta}\right) = \frac{u}{u_0}\frac{\partial}{\partial L}. \tag{3.7}$$

Equation (3.6) just says $ds_\gamma = \sqrt{2}\,u\,dT$, i.e., the distance travelled is the product of velocity and time. To check (3.7) consider a region

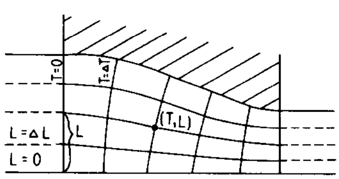

FIG. 2. (T, L)-coordinates: T denotes the time a particle has travelled since crossing line $T=0$ and L denotes the distance of a streamline from the center along line $T=0$.

$\Omega = \{(x, y) \mid 0 \leqslant L \leqslant L_1,\ \ 0 \leqslant T \leqslant T_1\}$. Denote $\partial\Omega_1 = \{(x, y) \mid 0 \leqslant L \leqslant L_1,$ $T = 0\}$ and $\partial\Omega_2 = \{(x, y) \mid 0 \leqslant L \leqslant L_1,\ T = T_1\}$. Then

$$\begin{aligned} &\int_\Omega \int \nabla \cdot \mathbf{u}\, dx\, dy = 0 \\ &\Leftrightarrow \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n}\, ds = 0 \\ &\Leftrightarrow \int_{\partial\Omega_1} \sqrt{2}\, u_0\, ds = \int_{\partial\Omega_2} \sqrt{2}\, u\, ds \\ &\Leftrightarrow \int_0^{L_1} u_0(L)\, dL = \int_0^{s_\delta(L_1)} u(T_1, L)\, dL. \end{aligned} \tag{3.8}$$

Here $s_\delta(L_1)$ denotes the arclength along the line $T = T_1$ from $L = 0$ to $L = L_1$. Differentiate with respect to L_1 and write L for L_1 to get

$$u_0\, dL = u\, ds_\delta. \tag{3.9}$$

Finally, we can write Eqs. (3.1)–(3.3) in characteristic coordinates:

$$\frac{\partial p}{\partial T} - \frac{\partial k}{\partial T} + 2k \frac{\partial(\ln u)}{\partial T} = 0 \tag{3.10}$$

$$\frac{\partial p}{\partial L} + \frac{\partial k}{\partial L} + 2k \frac{\partial(\ln u)}{\partial L} = 0 \tag{3.11}$$

$$\frac{\partial k}{\partial T} = h\left(2 \frac{\partial(\ln u)}{\partial T}\right). \tag{3.12}$$

The velocity equations (3.4), (3.5) now read

$$\frac{\partial u}{\partial T} = \frac{\sqrt{2}\, u^3}{u_0} \frac{\partial \phi}{\partial L} \tag{3.13}$$

$$\frac{\partial \phi}{\partial T} = \frac{\sqrt{2}\, u}{u_0} \frac{\partial u}{\partial L}. \tag{3.14}$$

The α- and β-sliplines are given by $T = \xi(L)$, where

$$\xi'(L) = \pm \frac{u_0}{\sqrt{2}\, u^2}. \tag{3.15}$$

4. Solving the Overdetermined System

The scheme for dealing with the overdetermined system is to eliminate p and k from the stress equations (3.10)–(3.12) and ϕ from the velocity equations (3.13)–(3.14). This leads to two higher-order partial differential equations for the velocity u, and we have to find solutions (if any exist) satisfying both equations. Luckily the first of these equations can be solved explicitly if h is a linear function. So, in the following assume a simple work-hardening rule

$$\frac{\partial k}{\partial T} = 2h \frac{\partial(\ln u)}{\partial T}, \tag{4.1}$$

where h is a constant.

Let us start by differentiating and solving for the mixed derivative $\partial^2 p/\partial L\, \partial T$ from (3.10) and (3.11). We get

$$\begin{aligned}
&\frac{\partial^2 k}{\partial L\, \partial T} - 2\frac{\partial k}{\partial L}\frac{\partial(\ln u)}{\partial T} - 2k\frac{\partial^2(\ln u)}{\partial L\, \partial T} \\
&\quad = -\frac{\partial^2 k}{\partial T\, \partial L} - 2\frac{\partial k}{\partial T}\frac{\partial(\ln u)}{\partial L} - 2k\frac{\partial^2(\ln u)}{\partial L\, \partial T} \\
&\quad \Rightarrow 2h\frac{\partial^2(\ln u)}{\partial L\, \partial T} - \frac{\partial k}{\partial L}\frac{\partial(\ln u)}{\partial T} + 2h\frac{\partial(\ln u)}{\partial T}\frac{\partial(\ln u)}{\partial L} = 0.
\end{aligned} \tag{4.2}$$

Solve this equation for $\partial k/\partial L$ and use (4.1) for $\partial k/\partial T$. Equating the mixed derivatives $\partial^2 k/\partial L\, \partial T$ from both equations leads after some cancellation to the simple equation

$$\frac{\partial^3(\ln u)}{\partial L\, \partial^2 T}\frac{\partial(\ln u)}{\partial T} - \frac{\partial^2(\ln u)}{\partial L\, \partial T}\frac{\partial^2(\ln u)}{\partial T^2} = 0. \tag{4.3}$$

Setting $\omega(T, L) = \partial(\ln u)/\partial T$ for further simplicity, we have

$$\omega\frac{\partial^2 \omega}{\partial T\, \partial L} - \frac{\partial \omega}{\partial L}\frac{\partial \omega}{\partial T} = 0. \tag{4.4}$$

The general solution of this non-linear partial differential equation is $\omega(T, L) = f_1(L)\, g(T)$, where $f_1(L)$ and $g(T)$ are arbitrary functions. This means

$$\frac{\partial(\ln u)}{\partial T}=f_1(L)\,g(T)$$

$$\Rightarrow \ln u=f_1(L)\int_0^T g(\tau)\,d\tau+f(L)$$

$$=f_1(L)\,G(T)+f(L)$$

$$\Rightarrow u(T,L)=e^{f(L)+f_1(L)\,G(T)} \qquad \text{where } G(0)=0. \tag{4.5}$$

For an arbitrary work-hardening modulus, this procedure can still be carried out but the resulting equation is significantly more complicated than (4.4) and cannot easily be solved explicitly for u.

Now turn to the velocity equations (3.13) and (3.14). Eliminating ϕ gives

$$\frac{\partial}{\partial T}\left(\frac{u_0}{\sqrt{2}\,u^3}\frac{\partial u}{\partial T}\right)=\frac{\partial}{\partial L}\left(\frac{\sqrt{2}\,u}{u_0}\frac{\partial u}{\partial L}\right)$$

$$\Leftrightarrow \frac{\partial^2(\ln u)}{\partial T^2}-2\left(\frac{\partial(\ln u)}{\partial T}\right)^2$$

$$=\frac{2u^4}{u_0^2}\left\{2\left(\frac{\partial(\ln u)}{\partial L}\right)^2+\frac{\partial^2(\ln u)}{\partial L^2}-\frac{\partial(\ln u_0)}{\partial L}\frac{\partial(\ln u)}{\partial L}\right\}. \tag{4.6}$$

The question now is: Are there solutions of the form (4.5) to Eq. (4.6)? Substituting we get the equation

$$f_1(L)\,G''(T)-2f_1(L)^2G'(T)^2$$

$$=2e^{2f(L)}e^{4f_1(L)\,G(T)}\{2f_1'(L)^2\,G(T)^2$$

$$+[f_1''(L)+3f_1'(L)f'(L)]\,G'(T)+f''(L)+f'(L)^2\}. \tag{4.7}$$

This equation may look rather discouraging. We have one equation for three unknown functions, f, f_1, and G, each depending on one of the two independent variables L or T. However, it is easy to check that the variables can be separated if we require

$$f_1(L)=1 \tag{4.8}$$

$$G''(T)-2G'(T)^2=2ce^{4G(T)} \tag{4.9}$$

$$f''(L)+f'(L)^2=ce^{-2f(L)} \tag{4.10}$$

where c can be any constant. In fact, in the Appendix it will be shown that Eq. (4.7) does not admit any other solutions. Thus all acceptable velocities are of the form $U(T,L)=e^{f(L)+G(T)}$ where G satisfies (4.9) with the boundary condition $G(0)=0$ and f satisfies (4.10). It is convenient to make a change of variables

$$y(L)=e^{f(L)}$$
$$z(T)=\tfrac{1}{2}e^{-2G(T)}.$$

Then the problem reduces into the form

$$u(T,L)=\frac{y(L)}{\sqrt{2z(T)}} \tag{4.11}$$

$$y''(L)\,y(L)=c \tag{4.12}$$

$$z''(T)\,z(T)=-c \tag{4.13}$$

$$z(0)=\tfrac{1}{2}. \tag{4.14}$$

In addition to these equations, for drawing processes we must have $z'(T)<0$ in order to make $u(T,L)$ increase along streamlines.

Before attempting to solve for y and z let us write ϕ, p, and k in terms of these functions. For ϕ we have

$$\frac{\partial\phi}{\partial L}=-\frac{1}{\sqrt{2}}\frac{z'(T)}{y(L)}$$

$$\frac{\partial\phi}{\partial T}=\frac{1}{\sqrt{2}}\frac{y'(L)}{z(T)}$$

$$\Rightarrow \phi(T,L)=\frac{-1}{\sqrt{2}\,c}z'(T)\,y'(L)+\phi_0, \tag{4.15}$$

where ϕ_0 is a constant.

This solution is only valid for $c\neq 0$. For $c=0$ z and y are both first-order polynomials. If $z'(T)\not\equiv 0$ and $y'(L)\not\equiv 0$ then

$$\phi(T,L)=\frac{1}{\sqrt{2}}\frac{y'(L)}{z'(T)}\ln z(T)-\frac{1}{\sqrt{2}}\frac{z'(T)}{y'(L)}\ln y(L)+\phi_0. \tag{4.16}$$

The special cases $z'(T)\equiv 0$ or $y'(L)\equiv 0$ give

$$z'(T)\equiv 0 \Rightarrow \phi(T,L)=\frac{1}{\sqrt{2}}\frac{y'(L)}{z(T)}T+\phi_0 \tag{4.17}$$

$$y'(L)\equiv 0 \Rightarrow \phi(T,L)=-\frac{1}{\sqrt{2}}\frac{z'(T)}{y(L)}L+\phi_0.$$

The equations for k and p can be integrated in a similar fashion and yield

$$k(T, L) = 2h \ln \frac{y(L)}{\sqrt{z(T)}} + k_0 \tag{4.18}$$

$$\begin{aligned} p(T, L) = 2h \ln y(L) \ln z(T) - \frac{h}{2} (\ln z(T))^2 \\ - 2h(\ln y(L))^2 + (k_0 - h) \ln z(T) \\ - 2(k_0 + h) \ln y(L) + p_0, \end{aligned} \tag{4.19}$$

where p_0 and k_0 are arbitrary constants. Notice that according to (4.18) k is proportional to the logarithm of the velocity.

5. Properties of the Velocity Field

Let us now return to equation

$$z''z = -c \tag{5.1}$$

and derive some helpful properties. Similar results apply naturally for y.

(1) The function z satisfying (5.1) can be viewed as a solution of a Hamiltonian system corresponding to a particle moving in a potential $v(z) = c \ln z$. Thus the equation can be integrated once. We have

$$z'' = -\frac{c}{z}$$

$$\Rightarrow \frac{z'^2}{2} + c \ln z = A. \tag{5.2}$$

Here A is the constant value of the Hamiltonian. Equation (5.2) can be used for numerical calculations.

(2) It is easy to check that if $z(T)$ is a solution of (5.1) corresponding to a fixed value of c, then $z_1(T) = \alpha z(T)$ and $z_2(T) = z(\alpha T)$ are both solutions corresponding to a constant $\alpha^2 c$. This scaling invariance makes it necessary only to solve (5.1) for values $c = 0$ and $c = \pm 1$. Other solutions are obtained by suitable scalings.

(3) As a consequence of 2, if z is a solution then $z_1(T) = (1/\alpha)\, z(\alpha T)$ is another solution corresponding to the same constant c. Initial conditions satisfy $z_1(0) = (1/\alpha)\, z(0)$ and $z_1'(0) = z'(0)$.

(4) Write (5.1) as a 2×2 system by defining $x_1(T)=z(T)$ and $x_2(T)=z'(T)$. The phase diagrams of the resulting system

$$x_1' = x_2$$
$$x_2' = -\frac{c}{x_1} \tag{5.3}$$

are shown in Fig. 3 for $c=\pm 1$ and $c=0$.
The qualitative behavior of functions z and y can be read from these graphs.

6. Some Die Profiles

By now the reader must be anxious to know what kind of die profile these solutions admit. Let us first study the solutions that can be solved explicitly, i.e., $c=0$. The general solutions for z and y are $y(L)=A+BL$ and $z(T)=\frac{1}{2}+DT$, where A, B, and D are constants. There are several subcases where the nature of the solution is different. Figure 4 shows the corresponding dies.

Case 1. $A=0$, $D<0$:

$$y(L)=BL$$
$$z(T)=\tfrac{1}{2}+DT$$
$$u(T,L)=\frac{BL}{\sqrt{1+2DT}}$$
$$\phi(T,L)=\frac{1}{\sqrt{2}}\frac{B}{D}\ln\left(\frac{1}{2}+DT\right)-\frac{1}{\sqrt{2}}\frac{D}{B}\ln BL+\phi_0. \tag{6.1}$$

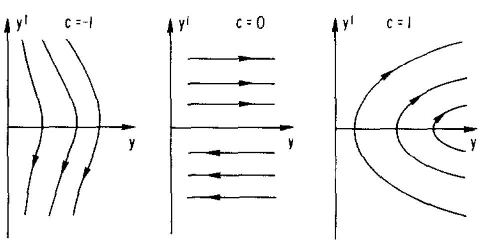

FIG. 3. Phase diagrams for the solution of the equation $yy''=c$ for different values of c.

This solution corresponds to the die found by Richmond (see Collins [1]). Streamlines are logarithmic spirals and α- and β-sliplines are radial lines and circular arcs.

Case 2. $B=0$, $D<0$:

$$\begin{aligned} y(L) &= A \\ z(T) &= \tfrac{1}{2} + DT \\ u(T, L) &= \frac{A}{\sqrt{1+2DT}} \\ \phi(T, L) &= -\frac{1}{\sqrt{2}}\frac{D}{A}L + \phi_0. \end{aligned} \tag{6.2}$$

In this solution ϕ is constant along streamlines which are consequently straight lines. The corresponding die is symmetric.

Case 3. $D=0$:

$$\begin{aligned} y(L) &= A + BL \\ z(T) &= \tfrac{1}{2} \\ u(T, L) &= A + BL \\ \phi(T, L) &= \sqrt{2}\, BT + \phi_0. \end{aligned} \tag{6.3}$$

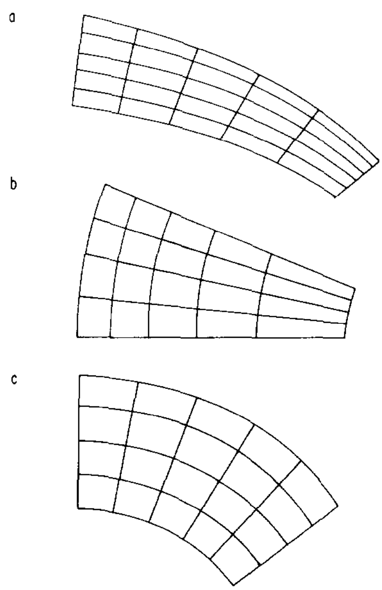

FIG. 4. Die profiles corresponding to $c=0$.

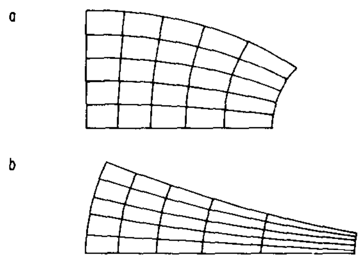

FIG. 5. Die profiles corresponding to $c = \pm 1$.

Here streamlines are circular lines and u and k are constants along them. The deformation corresponds to a rotation around a fixed point. There is no reduction in the diameter of the sheet going through this die.

For $c \neq 0$, finding the die profiles requires numerical calculations. However, it is easy to see that choosing $y'(0) = 0$ leads to a symmetric die. Figure 5 gives two die profiles corresponding to values $c = 1$ and $c = -1$.

What is the reduction in these dies? Let $D(T)$ be the diameter of the die measured along the line $T = \text{const}$. Then

$$\frac{D(T)}{D(0)} = \frac{\int_0^L (u_0/u)\, dl}{L} = \sqrt{2z(T)}. \tag{6.4}$$

7. BOUNDARY CONDITIONS

Let us now consider the question of joining the solutions to a uniform flow of rigid material on both sides of the die. Looking at the die profiles shown in the previous chapter one can anticipate difficulties. Indeed, it seems impossible to get a solution satisfying all natural conditions at the boundaries of the transforming region. This is in fact true as we will see in this section.

If $T = \xi(L)$ is the equation for the line where initial deformations take place, then at least the following equations should hold:

$$u(\xi(L), L) = \text{constant} \tag{7.1}$$

$$\phi(\xi(L), L) = \text{constant}. \tag{7.2}$$

Let us first study the case $c \neq 0$. Thus Eq. (4.15) can be used for ϕ.

Differentiating (7.1) and (7.2) with respect to L, we have

$$2y'(L)\, z(\xi(L)) - y(L)\, z'(\xi(L))\, \xi'(L) = 0 \tag{7.3}$$

$$y''(L)\, z'(\xi(L)) + y'(L)\, z''(\xi(L))\, \xi'(L) = 0. \tag{7.4}$$

For both equations to hold simultaneously, we must require

$$
\begin{aligned}
2y'(L)^2\, z(\xi(L))\, z''(\xi(L)) &= -z'(\xi(L))^2\, y(L)\, y''(L)\\
&\Rightarrow 2y'(L)^2 = z'(\xi(L))^2\\
&\Rightarrow \xi'(L) = \pm\sqrt{2}\,\frac{y''(L)}{z''(\xi(L))} = \pm\sqrt{2}\,\frac{z(\xi(L))}{y(L)}.
\end{aligned}
\tag{7.5}
$$

Comparing with (3.15) we observe that the entrance line must be an α- or β-slipline. This agrees with the results applying to other metal-forming processes. But can y and z be chosen so that $2y'^2 = z'^2$ holds on the boundary? For a symmetric die satisfying $y'(0) = 0$ this requires $z'(0) = 0$. So we must conclude the boundary conditions (7.1) and (7.2) can be satisfied in a die corresponding to $c = 1$ only at entrance and in one corresponding to $c = -1$ only at exit.

A similar analysis shows that in a spiral die (6.1) boundary conditions are satisfied at both sides if entrance and exit lines are characteristics, i.e., radial lines. A drawback in this case is that the corresponding die is not symmetric.

An appealing idea would be to form a die with $c = 1$ at the entrance region, possibly $c = 0$ in the middle and $c = -1$ at exit. However, it is possible to show that the equations do not allow for this kind of discontinuity. It seems that the best we can hope for is a die where the optimality condition $u = v$ is slightly violated in restricted regions.

Appendix

In this appendix, we attempt to show that the form of the solution found in Section 4 is the only possible one. Recall Eq. (4.7)

$$
\begin{aligned}
&f_1(L)\, G''(T) - 2f_1(L)^2\, G'(T)^2\\
&\quad = 2e^{2f(L)}e^{4f_1(L)\,G(T)}\{2f_1'(L)\, G(T)^2\\
&\qquad + [f_1''(L) + 3f_1'(L) f(L)]\, G(T) + f''(L) + f'(L)^2\},
\end{aligned}
\tag{A.1}
$$

where f, f_1, and G are unknown functions and $G(0) = 0$. To make the equation look a bit less frightening, denote

$$a_1 = 2f_1(L) \tag{A.2}$$

$$a_2 = \frac{4e^{2f(L)} f_1'(L)}{f_1(L)} \tag{A.3}$$

$$a_3 = \frac{2e^{2f(L)}[f_1''(L) + 3f_1'(L) f(L)]}{f_1(L)} \tag{A.4}$$

$$a_4 = \frac{2e^{2f(L)}[f''(L) + f'(L)^2]}{f_1(L)} \tag{A.5}$$

$$x_1 = G(T) \tag{A.6}$$

$$x_2 = G'(T). \tag{A.7}$$

Then we have

$$x_1' = x_2 \tag{A.8}$$

$$x_2' = a_1 x_3^2 + e^{2a_1 x_1}\{a_2 x_1^2 + a_3 x_1 + a_4\}, \tag{A.9}$$

where a_i's are functions of L and x_i's are functions of T. The scheme is to eliminate a_2, a_3, and a_4 from (A.9) using equations that we get by differentiating (A.9) with respect to T. For example, a_4 can be eliminated by writing (A.9) in the form

$$(x_2' - a_1 x_2^2)\, c^{-2a_1 x_1} = a_2 x_1^2 + a_3 x_1 + a_4$$

and differentiating it with respect to T. Division by x_1 and further differentiation eliminates a_3, etc. However, since this procedure has a drawback of not being valid if x_1 vanishes, let us carry out essentially the same elimination without divisions:

$$\begin{aligned} x_2'' &= 2a_1 x_2 x_2' + 2a_1 x_2 e^{2a_1 x_1}\{a_2 x_1^2 + a_3 x_1 + a_4\} \\ &\quad + x_2 e^{2a_1 x_1}\{2a_2 x_1 + a_3\} \\ &= 4a_1 x_2 x_2' - 2a_1^2 x_2^3 + x_2 e^{2a_1 x_1}\{2a_2 x_1 + a_3\} && \text{(A.10)} \\ \Rightarrow x_2''' &= 4a_1(x_2 x_2') + 6a_1^2 x_2^2 x_2' + (x_2' + 2a_1 x_2)\, e^{2a_1 x_1} \\ &\quad \times \{2a_2 x_1 + a_3\} + 2a_2 x_2^2 e^{2a_1 x_1} && \text{(A.11)} \\ \Rightarrow x_2 x_2''' &= 4a_1 x_2 (x_2 x_2')' + 6a_1^2 x_2^3 x_2^1 + (x_2' + 2a_1 x_2) \\ &\quad \times (x_2'' - 4a_1 x_2 x_2' + 2a_1^2 x_2^3) + 2a_2 x_2^2 e^{2a_1 x_1}. && \text{(A.12)} \end{aligned}$$

At this point the equations get rather long and tedious to write down. However, it is clear that a_2 can be eliminated by differentiating the last equation (A.12) once more with respect to T and using the resulting equation together with (A.12). The crucial thing to notice is that the resulting equation is a fourth-order polynomial of a_1. The coefficients are complicated functions of $x_2(T)$ but the exact form is irrelevant for what we want to do. So we can write

$$P_4(T)\,a_1^4(L)+P_3(T)\,a_1^3(L)+P_2(T)\,a_1^2(L)+P_1(T)\,a_1(L)+P_0(T)=0. \tag{A.13}$$

Now differentiate repeatedly with respect to L to get

$$\begin{aligned} &4P_4a_1^3+3P_3a_1^2+2p_2a_1+p_1=0 \quad \text{or} \quad a_1'=0\\ &\Rightarrow 12P_4a_1^2+6P_3a_1+2P_2=0 \quad \text{or} \quad a_1'=0\\ &\Rightarrow 24P_4a_1+6P_3=0 \quad \text{or} \quad a_1'=0. \end{aligned}$$

The last equation implies

$$a_1(L)=\frac{P_3(T)}{4P_4(T)},$$

and consequently $a_1(L)$ must be a constant. As a result $a_2(L)\equiv a_3(L)\equiv 0$. A simple separation of variables argument for Eq. (A.9) then requires that $a_4(L)=$ constant. There is no loss of generality setting $f_1(L)\equiv 1$, since any multiplying constant can be included into $G(T)$. We have reduced (A.1) to

$$f_1(L)=1 \tag{A.14}$$

$$f''(L)+f'(L)^2=ce^{2f(L)} \tag{A.15}$$

$$G''(T)-2G'(T)^2=2ce^{4G(T)} \tag{A.16}$$

for any constant c.

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